ON TWO-PARAMETER FELLER SEMIGROUP WITH NONLOCAL CONDITION FOR ONE-DIMENSIONAL DIFFUSION PROCESS

Bohdan KOPYTKO, Roman SHEVCHUK

Institute of Mathematics, Czestochowa University of Technology,
Czestochowa, Poland
Vasyl Stefanyk Precarpathian National University,
Ivano-Frankivsk, Ukraine

Ministerstwo Nauki
i Szkolnictwa Wyższego

**Organizacja IX Konferencji Modelowanie Matematyczne
w Fizyce i Technice (MMFT 2017)**
- zadanie finansowane w ramach umowy 829/P-DUN/2017
ze środków Ministra Nauki i Szkolnictwa Wyższego przeznaczonych
na działalność upowszechniającą naukę.

The report is devoted to the problem of construction of Feller semigroup associated with one-dimensional inhomogeneous diffusion process with membrane placed at the point, which location on real line is determined by the given function that depends on time variable. It is assumed that at the interior points of half-lines separated by membrane the desired process coincides with the ordinary diffusion processes given there and its behavior at the common boundary of these domains is described by the nonlocal Feller-Wentzell conjugation condition [1-3]. This problem is often called the problem of pasting together two diffusion processes on a line [4, 5].
The study of the problem is performed by analytical methods. Such an approach allows to determine the desired semigroup by means of the solution of the corresponding problem of conjugation for a linear parabolic equation of the second order (backward Kolmogorov equation) with discontinuous coefficients [4-6].

Consider on plane (s, x) the set

$$S_t = \{(s,x): \quad 0 \le s < t \le T, \ -\infty < x < \infty\},$$

denoting by $\overline{S}_t$ the closure of S_t. Suppose that $\overline{S}_t$ contains the continuous curve $x = h(s),\ 0 \le s \le T$, which separates S_t into two domains:

$$S_t^{(1)} = \{(s,x): \quad 0 \le s < t \le T,\ x < h(s)\}$$

and

$$S_t^{(2)} = \{(s,x): \quad 0 \le s < t \le T,\ x > h(s)\}.$$

Put $D_{1s} = (-\infty, h(s))$ and $D_{2s} = (X(s), \infty)$.
Consider in S_T two uniformly parabolic operators with bounded coefficients

$$\frac{\partial}{\partial s} + L_s^{(i)} \equiv \frac{\partial}{\partial s} + \frac{1}{2} b_i(s,x) \frac{\partial^2}{\partial x^2} + a_i(s,x) \frac{\partial}{\partial x}, \quad i = 1, 2. \qquad (1)$$

The problem is to find a solution $u(s, x, t)$ of equation

$$\frac{\partial u}{\partial s} + L_s^{(i)} u = 0, \quad (s, x) \in S_t^{(i)}, \ i = 1, 2, \tag{2}$$

which satisfies the 'initial' condition

$$\lim_{s \uparrow t} u(s, x, t) = \varphi(x), \quad x \in \mathbb{R}, \tag{3}$$

two conjugation conditions

$$u(s, h(s) - 0), t) = u(s, h(s) + 0, t), \quad 0 \le s \le t \le T, \tag{4}$$

$$\int_{D_{1s} \cup D_{2s}} [u(s, h(s), t) - u(s, y, t)] \mu(s, dy) = 0, \quad 0 \le s \le t \le T, \tag{5}$$

and two fitting conditions

$$\varphi(h(t) - 0) = \varphi(h(t) + 0), \tag{6}$$

$$\int_{D_{1t} \cup D_{2t}} [\varphi(h(t)) - \varphi(y)] \mu(t, dy) = 0. \tag{7}$$

If we assume that a solution $u(s,x,t)$ of problem (2)-(7) is the two-parameter Feller semigroup associated with some inhomogeneous Markov process on a line then the fulfilment for it the equation (2) points out that this process in D_{is} coincides with the diffusion processes given there generated by $L_s^{(i)}$, $i=1,2$, and the initial condition (3) is in agreement with the equality $T_{ss}=I$, where I is the identity operator. Next, the condition (4) reflects the Feller property of process and the equality (5) is the nonlocal Feller-Wentzell congugation condition which is responsible for jump-like exit of the process from the boundary of the domain.

The general Feller-Wentzell conjugation condition has the form

$$r(s)\frac{\partial u(s,X(s),t)}{\partial s}+q_1(s)\frac{\partial u(s,X(s)-0,t)}{\partial x}-q_2(s)\frac{\partial u(s,X(s)+0,t)}{\partial x}+$$
$$+\gamma(s)u(s,X(s),t)+\int\limits_{D_{1s}\cup D_{2s}}[u(s,X(s),t)-u(s,y,t)]\,\mu(s,dy)=0.$$

In our case $r(s)=q_1(s)=q_2(s)=\gamma(s)=0$ and $\mu(s,D_{1s}\cup D_{2s})=1$ for every $s\in[0,T]$.

I. The equation (2) is parabolic in $\overline{S}_T$, i.e., there exist positive constants b and B such that

$$0 < b \le b_i(s,x) \le B < \infty,\ i = 1,2,\ (s,x) \in \overline{S}_T.$$

II. In $\overline{S}_T$ the coefficients $b_i(s,x)$ and $a_i(s,x)$, $i = 1,2$, are continuous with respect to (s,x) and belong to $H^{\frac{\alpha}{2},\alpha}(\overline{S}_T)$, $0 < \alpha < 1$.

III. The initial function $\varphi(x)$ belongs to the space of bounded continuous functions which we denote by $C_b(\mathbb{R})$. The norm in this space is defined by the equality $\|\varphi\| = \sup\limits_{x\in\mathbb{R}} |\varphi(x)|$.

IV. The measure $\mu(s,\cdot)$ in (5) is nonnegative, $\mu(s, D_{1s} \cup D_{2s}) = 1$, $s \in [0,T]$, and for all $f \in C_b(\mathbb{R})$ the integrals

$$G_f^{(i)}(s) = \int\limits_{D_{is}} f(y)\mu(s,dy), \quad i = 1,2,$$

belong to $H^{\frac{1+\alpha}{2}}([0,T])$.

V. The curve $h(s)$ is continuous and belongs to $H^{\frac{1+\alpha}{2}}([0,T])$.

A solution of problem (2)-(7) has the form

$$u(s,x,t)=\int\limits_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy+$$
$$+\int\limits_s^t G_i(s,x,\tau,h(\tau))V_i(\tau,t)d\tau, \quad (s,x)\in \overline{S}_t^{(i)},\ i=1,2, \quad (8)$$

where G_i is the fundamental solution of the equation (2).
We denote by u_{i0} and u_{i1} the Poisson and the simple-layer potentials in (8) respectively. Then

$$u(s,x,t)=u_{i0}(s,x,t)+u_{i0}(s,x,t), \quad (s,x)\in \overline{S}_t^{(i)},\ i=1,2.$$

The unknown functions V_i are found from the conjugation conditions (4), (5) as a solution of the system of Volterra integral equations of the first kind

$$\int_s^t G_i(s,h(s),\tau,h(\tau))V_i(\tau,t)d\tau - \tag{9}$$

$$-\sum_{j=1}^{2}\int_s^t V_j(\tau,t)d\tau \int_{D_{js}} G_j(s,y,\tau,h(\tau))\mu(s,dy) = \Phi_i(s,t), \quad i=1,2,$$

where

$$\Phi_i(s,t) = \sum_{j=1}^{2}\int_{D_{js}} u_{j0}(s,y,t)\mu(s,dy) - u_{i0}(s,h(s),t), \quad i=1,2.$$

The function Φ_i, $i=1,2$, satisfies the inequality

$$|\Phi_i(s,t)-\Phi_i(\widetilde{s},t)| \le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}, \quad \widetilde{s}<s. \tag{10}$$

Regularization of equations of system (9) is performed by Holmgren transform. This transform is defined by integro-differential operator $\mathcal{E}$ which acts by the following rule

$$\mathcal{E}(s,t)\Phi_i = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int_s^t (\rho - s)^{-\frac{1}{2}}\Phi_i(\rho,t)d\rho, \quad 0 \le s < t \le T, \ i = 1,2.$$

Application of the operator $\mathcal{E}$ to both sides of each equation of system (9) gives the equivalent system of Volterra integral equations of the second kind of the form

$$V_i(s,t) = \sum_{j=1}^{2}\int_s^t K_{ij}(s,\tau)V_j(\tau,t)d\tau + \Psi_i(s,t), \quad i = 1,2. \tag{11}$$

The system of Volterra integral equations of the second kind

$$\Psi_i(s,t) = \sqrt{\frac{b_i(s,h(s))}{2\pi}} \int\limits_s^t (\rho - s)^{-\frac{3}{2}} [\Phi_i(s,t) - \Phi_i(\rho,t)] d\rho +$$

$$+ \sqrt{\frac{2b_i(s,h(s))}{\pi}} (t-s)^{-\frac{1}{2}} \Phi_i(s,t), \quad i = 1,2,$$

$$K_{ij}(s,\tau) = \sqrt{\frac{2b_i(s,h(s))}{\pi}} \cdot \frac{\partial}{\partial s} N_{ij}(s,\tau), \quad i = 1,2, \ j = 1,2,$$

$$N_{ii}(s,\tau) = \int\limits_s^\tau (\rho - s)^{-\frac{1}{2}} \Big[(Z_{i0}(\rho,h(\rho),\tau,h(\tau)) - Z_{i0}(\rho,0,\tau,0)) +$$

$$+ Z_{i1}(\rho,h(\rho),\tau,h(\tau)) - \int\limits_{D_{i\rho}} G_i(\rho,y,\tau,h(\tau)) \mu(\rho,dy) \Big] d\rho, \quad i = j,$$

$$N_{ij}(s,\tau) = -\int\limits_s^\tau (\rho - s)^{-\frac{1}{2}} d\rho \int\limits_{D_{j\rho}} G_j(\rho,y,\tau,h(\tau)) \mu(\rho,dy), \quad i \neq j.$$

The estimation for function Ψ_i:

$$|\Psi_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad 0 \le s < t \le T. \tag{12}$$

The representation of kernel K_{ij}:

$$K_{ij}(s,\tau) = K_{ij}^{(1)}(s,\tau) + K_{ij}^{(2)}(s,\tau), \tag{13}$$

where

$$|K_{ij}^{(1)}(s,\tau)| \le C(\delta)(\tau-s)^{-1+\frac{\alpha}{2}},$$

$$K_{ij}^{(2)}(s,\tau) = \sqrt{b_i(s,h(s))b_j(\tau,h(\tau))} \int\limits_{D_{js}^{\delta}} \frac{\partial Z_{j0}(s,y,\tau,h(\tau))}{\partial y}\mu(s,dy)$$

$(D_{js}^{\delta} = \{y \in D_{js} : \ |y-h(s)| < \delta\}, \ \delta$ is any positive number.)

Note that the function $K_{ij}^{(2)}$ has a 'strong' singularity:

$$|K_{ij}^{(2)}(s,\tau)| \le C(\delta)(\tau-s)^{-1}.$$

Despite a 'strong' singularity of kernels $K_{ij}(s,\tau)$, we can apply to the system of equations (11) the ordinary method of successive approximations. We find the function $V_i(s,t)$ of the form

$$V_i(s,t) = \sum_{n=0}^{\infty} V_i^{(n)}(s,t), \quad 0 \le s < t \le T, \ i = 1, 2, \tag{14}$$

where

$$V_i^{(0)}(s,t) = \Psi_i(s,t),$$
$$V_i^{(n)}(s,t) = \sum_{j=1}^{2} \int_s^t K_{ij}(s,\tau) V_i^{(n-1)}(\tau,t)d\tau, \quad n = 1, 2, \ldots$$

The convergence of series (14) follows from the next inequality, which is established by the method of mathematical induction:

$$\left|V_i^{(n)}(s,t)\right| \le C\|\varphi\|(t-s)^{-\frac{1}{2}} \sum_{k=0}^{n} C_n^k a^{(n-k)}(m(\delta))^k, \quad n=0,1,\dots, \tag{15}$$

where

$$a^{(k)} = \frac{\left(2c(\delta)T^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\right)^k \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+k\alpha}{2}\right)}, \quad k=0,1,\dots,n,$$

$$m(\delta) = \max_{s\in[0,T]} \mu(s, D_{1s}^{\delta} \cup D_{2s}^{\delta}) < 1 \text{ (for sufficiently small } \delta).$$

From the estimation (15) it also follows that

$$|V_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad 0 \le s < t \le T. \tag{16}$$

Theorem 1

Suppose that the conditions I-V are satisfied. Then for any function $\varphi \in C_b(\mathbb{R})$ there exist the unique solution $u(s,x,t)$ of parabolic conjugation problem (2)-(7) from class $C^{1,2}(S_t^{(1)} \cup S_t^{(2)}) \cap C(\overline{S}_t)$. Furthermore, this solution is expressed by formulas (8), (14) and for it the estimation

$$|u(s,x,t)| \leq C\|\varphi\|, \quad 0 \leq s \leq t \leq T,\ x \in \mathbb{R} \tag{17}$$

holds.

Define the two-parameter family of linear operators $T_{st},\ 0 \leq s < t \leq T$, acting on the function $\varphi \in C_b(\mathbb{R})$ by the rule:

$$T_{st}\varphi(x) = u(s,x,t), \tag{18}$$

where $u(s,x,t)$ is the solution of problem (2)-(7), which is expressed by formulas (8), (14).

Operators T_{st} have the following properties:

- if $\varphi \in C_b(\mathbb{R})$ and $\varphi(x) \geq 0$ for all $x \in \mathbb{R}$, then $T_{st}\varphi(x) \geq 0$ for all $0 \leq s < t \leq T,\ x \in \mathbb{R}$;
- T_{st} are contractive operators, i.e., they do not increase the norm of element;
- $T_{st} = T_{s\tau}T_{\tau t},\ 0 \leq s < \tau < t \leq T$ (the semigroup property);
- if the sequence $\varphi_n \in C_b(\mathbb{R})$ such that $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}$ and, besides, $\sup_n \|\varphi_n\| < \infty$, then $\lim\limits_{n\to\infty} T_{st}\varphi_n(x) = T_{st}\varphi(x)$ for all $0 \leq s < t \leq T,\ x \in \mathbb{R}$.

Theorem 2

Let the conditions of Theorem 1 hold. Then the two-parameter family of operators $T_{st},\ 0 \leq s \leq t \leq T$, constructed by formula (18), describes the inhomogeneous Feller process on $\mathbb{R}$ such that in domains D_{1s} and D_{2s} it coincides with the diffusion processes given there generated by $L_s^{(i)},\ i = 1, 2$, and its behavior after reaching the common boundary of these domains $h(s)$ is determined by Feller-Wentzell conjugation condition (5).

[1] Feller W., The parabolic differential equations and associated semigroups of transformations, Ann. of Math. Soc. 1952, 55, 468-519.

[2] Wentzell A.D., Semigroups of operators that correspond to a generalized differential operator of second order, Dokl. AN SSSR 1956, 111, 2, 269-272.

[3] Langer H., Schenk W., Knotting of one-dimensional Feller process, Math. Nachr. 1983, 113, 151-161.

[4] Portenko M.I., Diffusion Processes in Media with Membranes, Institute of Mathematics of the NAS of Ukraine, Kyiv 1995.

[5] Kopytko B.I., Sewing two nonhomogeneous diffusion processes on a straight line, Ukrainian Math. Journal 1983, 35, 2, 135-141.

[6] Shevchuk R.V., Pasting of two one-dimensional diffusion processes, Annales Mathematicae et Informaticae 2012, 39, 225-239.

[7] Pogorzelski W., Etude de la solution fondamentale de l'?quation parabolique, Richerce Matem. 1956, 5, 25-57.

[8] Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.N., Linear and Quasilinear Equations of Parabolic Type, Nauka, Moscow 1967.

[9] Kamynin L.I., A boundary value problem in the theory of heat conduction with a nonclassical boundary condition, Comput. Math. Math. Phys. 1964, 4, 6, 33-59.

[10] Skubachevskii A.L., On Feller semigroups for multidimensional diffusion processes, Dokl. AN 1995, 341, 2, 173-176.

[11] Pilipenko A.Yu., On the Skorokhod mapping for equations with reflection and possible jumplike exit from a boundary, Ukrainian Math. J. 2012, 63, 9, 1415-1432.

DZIĘKUJĘ ZA UWAGĘ!